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Symmetric reduction of the vectorial fundamental transformation: application to the Darboux–Egorov equations

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Abstract. The vectorial fundamental transformation for the Darboux equations is reduced to the symmetric case. This is combined with the orthogonal reduction of Lamé type to obtain reductions of the vectorial Ribaucour transformations to Egorov systems. We also show that a permutability property holds for all these transformations. As an example we apply these transformations to the Cartesian background.

1. Introduction

At the turn of this century a number of results on differential geometry were already well established [1, 3, 4, 10]. We are thinking of conjugate nets described by the Darboux equations [3, 10] and related transformations [12] of Laplace, Lévy [16] and fundamental [11, 14] types; and the orthogonal nets, described by the Lamé equations [4, 15], and their Ribaucour transformations [4, 22]. The Lamé equations describe flat diagonal metrics, among which we find a distinguished class: those of Egorov type. These particular classes of flat diagonal metrics are described by the Darboux–Egorov equations [4], that were first proposed by Darboux [5] and studied further in [13, 21, 23] and finally, as was recognized by Darboux [4], Egorov gave an almost definitive treatment in [9].

The mentioned results are deeply connected with the modern theory of integrable systems. It is well known that integrable equations such as Liouville or sine–Gordon were first considered in the context of differential geometry: minimal and pseudo-spherical surfaces. It has been discovered recently that the N-component Kadomtsev–Petviashvili (KP) hierarchy describes the iso-conjugate transformations of the Darboux equations, and its vertex operators correspond precisely to Laplace, Lévy and fundamental transformations [8]. Moreover, the N-component BKP hierarchy models the iso-orthogonal deformations of the Lamé equations, its vertex operator being the Ribaucour transformation [20].

In previous papers we have considered the iteration of the mentioned transformations within modern soliton theory: in [18] we studied the Lévy transformation and in [17] the Ribaucour transformation, using a vectorial approach in the latter. In this paper we give a reduction of the vectorial fundamental transformation [7, 19] to the symmetric case, and further to the Darboux–Egorov equations.

The layout of this paper is as follows. In section 2 we consider the symmetric reduction of the Darboux equations obtaining the vectorial symmetric fundamental transformations for all

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the geometrical data; next, in section 3, we combine these results with those of [17] to get the reduction of the vectorial Ribaucour to the Darboux–Egorov equations, giving the expressions of all the transformed relevant geometrical data. In both sections we consider the dressing of the Cartesian background and prove that the permutability property is preserved under all the reductions considered. We must recall that the permutability property of these transformations is an important issue that for the fundamental transformation was considered in [7, 11, 14] and for the Ribaucour transformation in [1, 2, 6, 17].

2. The vectorial fundamental transformation for the symmetric Darboux equations

The Darboux equations

$$\frac{\partial\beta_{ij}}{\partial u_k} - \beta_{ik}\beta_{kj} = 0 \qquad i, j, k = 1, \ldots, N \quad \text{with } i, j, k \text{ different} \tag{1}$$

for the $N(N-1)$ functions $\{\beta_{ij}\}_{i,j=1,\ldots,N,\, i\neq j}$ of $\boldsymbol{u} := (u_1, \ldots, u_N)$, characterize N-dimensional submanifolds of $\mathbb{R}^D$, $N \leqslant D$, parametrized by conjugate coordinate systems [3, 10], and are the compatibility conditions of the following linear system:

$$\frac{\partial \boldsymbol{X}_j}{\partial u_i} = \beta_{ji}\boldsymbol{X}_i \qquad i, j = 1, \ldots, N \quad i \neq j \tag{2}$$

involving suitable D-dimensional vectors $\boldsymbol{X}_i$, tangent to the coordinate lines.

The so-called Lamé coefficients satisfy

$$\frac{\partial H_j}{\partial u_i} = \beta_{ij}H_i \qquad i, j = 1, \ldots, N \quad i \neq j \tag{3}$$

and the points of the surface $\boldsymbol{x} = (x_1, \ldots, x_N)$ can be found by means of

$$\frac{\partial \boldsymbol{x}}{\partial u_i} = \boldsymbol{X}_i H_i \qquad i = 1, \ldots, N. \tag{4}$$

The fundamental transformation for the Darboux system was introduced in [11, 14], and its vectorial extension was given in [7, 19]. It requires the introduction of a potential in the following manner: given vector solutions $\boldsymbol{\xi}_i \in V$ and $\boldsymbol{\zeta}_i^* \in W^*$ of (2) and (3), $i = 1, \ldots, N$, respectively, where V, W are linear spaces and W^* is the dual space of W, one can define a potential matrix $\Omega(\boldsymbol{\xi}, \boldsymbol{\zeta}^*)$: $W \to V$ through the equations

$$\frac{\partial\Omega(\boldsymbol{\xi}, \boldsymbol{\zeta}^*)}{\partial u_i} = \boldsymbol{\xi}_i \otimes \boldsymbol{\zeta}_i^*. \tag{5}$$

Given solutions $\boldsymbol{\xi}_i \in V$ and $\boldsymbol{\xi}_i^* \in V^*$ of (2) and (3), $i = 1, \ldots, N$, respectively, new rotation coefficients $\hat{\beta}_{ij}$, tangent vectors $\hat{\boldsymbol{X}}_i$, Lamé coefficients $\hat{H}_i$ and points of the surface $\hat{\boldsymbol{x}}$ are given by

$$\begin{aligned}
\hat{\beta}_{ij} &= \beta_{ij} - \langle \boldsymbol{\xi}_j^*, \Omega(\boldsymbol{\xi}, \boldsymbol{\xi}^*)^{-1}\boldsymbol{\xi}_i\rangle \\
\hat{\boldsymbol{X}}_i &= \boldsymbol{X}_i - \Omega(\boldsymbol{X}, \boldsymbol{\xi}^*)\,\Omega(\boldsymbol{\xi}, \boldsymbol{\xi}^*)^{-1}\boldsymbol{\xi}_i \\
\hat{H}_i &= H_i - \boldsymbol{\xi}_i^*\Omega(\boldsymbol{\xi}, \boldsymbol{\xi}^*)^{-1}\Omega(\boldsymbol{\xi}, H) \\
\hat{\boldsymbol{x}} &= \boldsymbol{x} - \Omega(\boldsymbol{X}, \boldsymbol{\xi}^*)\,\Omega(\boldsymbol{\xi}, \boldsymbol{\xi}^*)^{-1}\Omega(\boldsymbol{\xi}, H).
\end{aligned} \tag{6}$$

Here we assume that $\Omega(\boldsymbol{\xi}, \boldsymbol{\xi}^*)$ is invertible. We shall refer to this transformation as the vectorial fundamental transformation with transformation data $(V, \boldsymbol{\xi}_i, \boldsymbol{\xi}_i^*)$.

The symmetric reduction we are concerned with requires the rotation coefficients to be symmetric; i.e.

$$\begin{aligned} &\frac{\partial \beta_{ij}}{\partial u_k} - \beta_{ik}\beta_{kj} = 0 \qquad && i, j, k = 1, \ldots, N \quad \text{with } i, j, k \text{ different} \\ &\beta_{ij} - \beta_{ji} = 0 && i, j = 1, \ldots, N \qquad i \neq j. \end{aligned} \tag{7}$$

We now consider which transformation data $(V, \boldsymbol{\xi}_i, \boldsymbol{\xi}_i^*)$ gives a vectorial fundamental transformation that preserves the symmetric Darboux equations (7).

We make the following observations.

(a) Given a solution $\boldsymbol{\xi}_i \in V$ of (2) then

$$\boldsymbol{\xi}_i^* := \boldsymbol{\xi}_i^{\mathrm{t}} L \tag{8}$$

where $^{\mathrm{t}}$ means transpose and $L \in \mathrm{L}(V)$ is a linear operator on V, is a V^*-valued solution of (3) if and only if (7) holds. We shall say that L is the associated linear operator.

(b) Given symmetric β's, $\boldsymbol{\xi}_i \in V$ and $\boldsymbol{\zeta}_i \in W$ solutions of (2) and $\boldsymbol{\xi}_i^*$ and $\boldsymbol{\zeta}_i^*$ as prescribed in (8), $i = 1, \ldots, N$, with associated linear operators L and M, respectively; then

$$\frac{\partial}{\partial u_i}(L^{\mathrm{t}}\Omega(\boldsymbol{\xi}, \boldsymbol{\zeta}^*) - \Omega(\boldsymbol{\zeta}, \boldsymbol{\xi}^*)^{\mathrm{t}} M) = 0 \qquad i = 1, \ldots, N.$$

(c) Suppose given a solution β_{ij} of the symmetric Darboux equations (7), $\boldsymbol{\xi}_i \in V$ and $\boldsymbol{\zeta}_i \in W$ solving (2) and $\boldsymbol{\xi}_i^*$ and $\boldsymbol{\zeta}_i^*$ as prescribed in (8), with associated linear operators L and M, respectively. Then, if

$$\begin{aligned} &L^{\mathrm{t}}\Omega(\boldsymbol{\xi}, \boldsymbol{\zeta}^*) - \Omega(\boldsymbol{\zeta}, \boldsymbol{\xi}^*)^{\mathrm{t}} M = 0 \\ &L^{\mathrm{t}}\Omega(\boldsymbol{\xi}, \boldsymbol{\xi}^*) - \Omega(\boldsymbol{\xi}, \boldsymbol{\xi}^*)^{\mathrm{t}} L = 0 \end{aligned} \tag{9}$$

the vectorial fundamental transformation (6):

$$\begin{aligned} &\hat{\beta}_{ij} = \beta_{ij} - \langle \boldsymbol{\xi}_j^*, \Omega(\boldsymbol{\xi}, \boldsymbol{\xi}^*)^{-1}\boldsymbol{\xi}_i \rangle \\ &\hat{\boldsymbol{\zeta}}_i = \boldsymbol{\zeta}_i - \Omega(\boldsymbol{\zeta}, \boldsymbol{\xi}^*)\,\Omega(\boldsymbol{\xi}, \boldsymbol{\xi}^*)^{-1}\boldsymbol{\xi}_i \\ &\hat{\boldsymbol{\zeta}}_i^* = \boldsymbol{\zeta}_i^* - \boldsymbol{\xi}_i^*\Omega(\boldsymbol{\xi}, \boldsymbol{\xi}^*)^{-1}\Omega(\boldsymbol{\xi}, \boldsymbol{\zeta}^*) \end{aligned}$$

is such that

$$\hat{\boldsymbol{\zeta}}_i^* := \hat{\boldsymbol{\zeta}}_i^{\mathrm{t}} M.$$

Therefore, as $\Omega(\boldsymbol{\xi}, \boldsymbol{\zeta}^*)$ and $\Omega(\boldsymbol{\zeta}, \boldsymbol{\xi}^*)$ are defined by (5) up to additive constant matrices, we may take them such that

$$L^{\mathrm{t}}\Omega(\boldsymbol{\xi}, \boldsymbol{\zeta}^*) - \Omega(\boldsymbol{\zeta}, \boldsymbol{\xi}^*)^{\mathrm{t}} M = 0.$$

These observations imply one of the main results of this paper.

Theorem. *The vectorial fundamental transformation (6) preserves the symmetric Darboux equations whenever the transformation data $(V, \boldsymbol{\xi}_i, \boldsymbol{\xi}_i^*)$ satisfies*

$$\begin{aligned} &\boldsymbol{\xi}_i^* = \boldsymbol{\xi}_i^{\mathrm{t}} L \\ &L^{\mathrm{t}}\Omega(\boldsymbol{\xi}, \boldsymbol{\xi}^*) - \Omega(\boldsymbol{\xi}, \boldsymbol{\xi}^*)^{\mathrm{t}} L = 0. \end{aligned}$$

We say that $(V, \boldsymbol{\xi}_i, L)$ is the transformation data for this particular vectorial fundamental transformation that we call the vectorial symmetric fundamental transformation.

2.1. Permutability of vectorial symmetric fundamental transformations

The vectorial fundamental transformations permute among them; i.e. the composition of two vectorial fundamental transformations yields a new vectorial fundamental transformation. When these two transformations are done in different order the resulting composed vectorial fundamental transformation is equivalent, through conjugation by a permutation matrix, to the first composed vectorial fundamental transformation, so that all the geometrical data are identical for both composed transformations; hence, the permutability character of these transformations. Moreover, it also follows that the vectorial fundamental transformation is just a superposition of a number of fundamental transformations.

One can easily conclude that this result can be extended to the vectorial symmetric fundamental transformation:

Proposition. *The vectorial symmetric fundamental transformation with transformation data*

$$\left(V_1 \oplus V_2, \begin{pmatrix} \boldsymbol{\xi}_{i,(1)} \\ \boldsymbol{\xi}_{i,(2)} \end{pmatrix}, \begin{pmatrix} L_{(1)} & 0 \\ 0 & L_{(2)} \end{pmatrix}\right)$$

coincides with the following composition of vectorial symmetric fundamental transformations.

(a) First transform with data

$$(V_2, \boldsymbol{\xi}_{i,(2)}, L_{(2)})$$

and denote the transformation by $'$.

(b) On the result of this transformation apply a second one with data

$$(V_1, \boldsymbol{\xi}'_{i,(1)}, L_{(2)}).$$

Proof. Because the transformation data follows the prescription of our theorem it must satisfy

$$\begin{aligned} &(\boldsymbol{\xi}^*_{i,(s)})^{\mathrm{t}} = L_{(s)}\boldsymbol{\xi}_{i,(s)} && s = 1, 2 \\ &L^{\mathrm{t}}_{(s)}\Omega(\boldsymbol{\xi}_{(s)}, \boldsymbol{\xi}^*_{(s)}) - \Omega(\boldsymbol{\xi}_{(s)}, \boldsymbol{\xi}^*_{(s)})^{\mathrm{t}} L_{(s)} = 0 && s = 1, 2 \\ &L^{\mathrm{t}}_{(1)}\Omega(\boldsymbol{\xi}_{(1)}, \boldsymbol{\xi}^*_{(2)}) - \Omega(\boldsymbol{\xi}_{(2)}, \boldsymbol{\xi}^*_{(1)})^{\mathrm{t}} L_{(2)} = 0. \end{aligned}$$

The first vectorial fundamental transformation is a vectorial symmetric one with data $(V, \boldsymbol{\xi}_{i,(2)}, L_{(2)})$. Observation (c) implies that the vectorial fundamental transformation of point (b) is also a vectorial symmetric fundamental transformation. □

From these results we conclude that the composition of scalar symmetric fundamental transformations results in a vectorial symmetric fundamental transformation with associated matrix of diagonal type. In fact, when the associated matrix L is not diagonal the corresponding transformation cannot be obtained by means of composition only.

Dressing the Cartesian background. The Cartesian net has $\boldsymbol{X}_i = \boldsymbol{e}_i$, with $\{\boldsymbol{e}_i\}_{i=1,\dots,N}$ a linear independent set of vectors of $\mathbb{R}^D$, $H_i = 1$, the coordinates are $\boldsymbol{x}(\boldsymbol{u}) = \boldsymbol{u}$ and vanishing rotation coefficients $\beta_{ij} = 0$, and hence $\boldsymbol{\xi}_i = \boldsymbol{\xi}_i(u_i)$. The points of the new symmetric net are given by

$$\boldsymbol{x}(\boldsymbol{u}) = \boldsymbol{u} - \left[A + \sum_{i=1,\dots,N} \boldsymbol{e}_i \otimes \int_{u_{i,0}}^{u_i} \mathrm{d}u_i\, \boldsymbol{\xi}^{\mathrm{t}}_i(u_i)\right]\left[\sum_{i=1,\dots,N} \Omega_i(u_i)\right]^{-1} \times \left[\boldsymbol{c} + \sum_{i=1,\dots,N} \int_{u_{i,0}}^{u_i} \mathrm{d}u_i\, \boldsymbol{\xi}_i(u_i)\right]$$

where A is a constant $D \times M$ matrix, $\boldsymbol{c} \in \mathbb{R}^M$, $\Omega_i(u_i) = \int_{u_{i,0}}^{u_i} \mathrm{d}u_i\, \boldsymbol{\xi}_i \otimes \boldsymbol{\xi}^{\mathrm{t}}_i L + \Omega_{i,0}$ with $L^{\mathrm{t}}\Omega_{i,0} - \Omega^{\mathrm{t}}_{i,0} L = 0$.

3. Vectorial transformation for the Egorov metrics

The Lamé equations describe N-dimensional conjugate orthogonal systems of coordinates [4, 15, 24]:

$$\frac{\partial\beta_{ij}}{\partial u_k} - \beta_{ik}\beta_{kj} = 0 \qquad i, j, k = 1, \dots, N \quad \text{with } i, j, k \text{ different} \tag{10}$$

$$\frac{\partial\beta_{ij}}{\partial u_i} + \frac{\partial\beta_{ji}}{\partial u_j} + \sum_{\substack{k=1,\dots,N\\ k\neq i,j}} \beta_{ki}\beta_{kj} = 0 \qquad i, j = 1, \dots, N \quad i \neq j. \tag{11}$$

Now we have orthogonal tangent directions, $\boldsymbol{X}_i \cdot \boldsymbol{X}_j = \delta_{ij}$.

The reduction of the vectorial fundamental transformation to the orthogonal case; i.e. the vectorial Ribaucour transformation, was studied by us in [17]. The symmetric reduction of the Darboux equations can be combined with the Lamé equations to obtain the so-called equivalent system of Darboux–Egorov equations

$$\begin{aligned}
&\frac{\partial\beta_{ij}}{\partial u_k} - \beta_{ik}\beta_{kj} = 0 && i, j, k = 1, \dots, N \quad \text{with } i, j, k \text{ different}\\
&\beta_{ij} - \beta_{ji} = 0 && i, j = 1, \dots, N \quad i \neq j\\
&\sum_{k=1}^{N} \frac{\partial\beta_{ij}}{\partial u_k} = 0 && i, j = 1, \dots, N \quad i \neq j.
\end{aligned} \tag{12}$$

The reduction of the vectorial fundamental transformation to the Darboux–Egorov case can be thought of as a superposition of two reductions, namely the symmetric together with the orthogonal reduction. Thus, we must request to the transforming data and potential, the constraints for both reductions. This implies the second main result of this paper.

Theorem. *The vectorial fundamental transformation (6) preserves the Darboux–Egorov equations (12) whenever the transformation data (V, $\boldsymbol{\xi}_i$, $\boldsymbol{\xi}_i^*$) satisfy*

$$\boldsymbol{\xi}_i^* = \boldsymbol{\xi}_i^{\mathrm{t}} L = \left(\frac{\partial\boldsymbol{\xi}_i}{\partial u_i} + \sum_{\substack{k\neq i\\ k=1,\dots,N}} \boldsymbol{\xi}_k \beta_{ki} \right)^{\mathrm{t}}$$

$$L^{\mathrm{t}}\Omega(\boldsymbol{\xi}, \boldsymbol{\xi}^*) - \Omega(\boldsymbol{\xi}, \boldsymbol{\xi}^*)^{\mathrm{t}} L = 0$$

$$\Omega(\boldsymbol{\xi}, \boldsymbol{\xi}^*) + \Omega(\boldsymbol{\xi}, \boldsymbol{\xi}^*)^{\mathrm{t}} = \sum_{k=1,\dots,N} \boldsymbol{\xi}_k \otimes \boldsymbol{\xi}_k^{\mathrm{t}}$$

for some linear operator $L \in \mathrm{L}(V)$.

Observe that in the scalar case, $M = 1$, the second equation above is trivial and the third one determines the potential completely. The associated transformation in this case can be found in [24].

Permutability. The vectorial Ribaucour transformation was shown to have the permutability property in [17], moreover it was carried out as in our proof of the permutability for the symmetric case. This implies that the combination of both reductions should share the permutability character of the symmetric and orthogonal reduction.

Dressing the Cartesian background. Now we have $\boldsymbol{\xi}_i = \exp(L^{\mathrm{t}} u_i)\boldsymbol{a}_i$ with $\boldsymbol{a}_i \in \mathbb{R}^M$ constant vectors. In the diagonal case, $L = \operatorname{diag}(\ell_1, \ldots, \ell_N)$ we find

$$\Omega(\boldsymbol{\xi}, \boldsymbol{\xi}^*) = (\Omega_{ij}) \qquad \Omega_{ij} := \frac{\ell_j}{\ell_i + \ell_j} \sum_{k=1}^{N} \exp((\ell_i + \ell_j)u_k) a_{k,i} a_{k,j}$$

$$H_i(\boldsymbol{u}) = 1 - \frac{1}{|\Omega(\boldsymbol{u})|} \sum_{k,l=1}^{M} a_{i,k} \exp(\ell_k u_i) \ell_k \operatorname{cofac}(\Omega(\boldsymbol{u}))_{kl} \left(c_l + \frac{1}{\ell_l} \sum_{j=1}^{N} \exp(\ell_l u_j) a_{j,l} \right)$$

$$x_i(\boldsymbol{u}) = u_i - \frac{1}{|\Omega(\boldsymbol{u})|} \sum_{k,l=1}^{M} a_{i,k} \exp(\ell_k u_i) \operatorname{cofac}(\Omega(\boldsymbol{u}))_{kl} \left(c_l + \frac{1}{\ell_l} \sum_{j=1}^{N} \exp(\ell_l u_j) a_{j,l} \right)$$

$$\beta_{ij}(\boldsymbol{u}) = -\frac{1}{|\Omega(\boldsymbol{u})|} \sum_{k,l=1}^{M} \ell_k a_{j,k} \operatorname{cofac}(\Omega(\boldsymbol{u}))_{kl} a_{i,l} \exp(\ell_k u_j + \ell_l u_i)$$

where $\boldsymbol{a}_l^{\mathrm{t}} := (a_{l,1}, \ldots, a_{l,N})$, and we are using the cofactor matrix $\operatorname{cofac}(A)$; i.e. $A^{-1} = |A|^{-1} \operatorname{cofac} A$. These type of solutions are the extension to multidimensions of the bright multi-soliton solutions of the attractive nonlinear Schrödinger equation, which describes the propagation of optical pulses in nonlinear fibres.

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References

[1] Bianchi L 1904 *Rendi. Lincei* **13** 361
[2] Bianchi L 1924 *Lezioni di Geometria Differenziale* 3rd edn (Bologna: Zanichelli)
[3] Darboux G 1896 *Leçons sur la Théorie Générale des Surfaces IV* (Paris: Gauthier-Villars) (reprinted 1972 (New York: Chelsea))
[4] Darboux G 1910 *Leçons sur les Systèmes Orthogonaux et les Coordenées Curvilignes* 2nd edn (Paris: Gauthier-Villars) (the first edition was in 1897; reprinted 1993 (Sceaux: Éditions Jacques Gabay))
[5] Darboux G 1866 *Ann. L École Normale* **3** 97
[6] Demoulin M A 1910 *Comp. Rend. Acad. Sci., Paris* **150** 156
[7] Doliwa A, Santini P M and Mañas M 1999 Transformations for quadrilateral lattices *J. Math. Phys.* at press (Doliwa A, Santini P M and Mañas M 1997 *Preprint* solv-int/9712017)
[8] Doliwa A, Mañas M, Martínez Alonso L, Medina E and Santini P M 1999 *J. Phys. A: Math. Gen.* **32** 1197
[9] Egorov D-Th 1900 *Comp. Rend. Acad. Sci., Paris* **131** 668
Egorov D-Th 1901 *Comp. Rend. Acad. Sci., Paris* **132** 174
[10] Eisenhart L P 1909 *A Treatise on the Differential Geometry of Curves and Surfaces* (Boston, MA: Ginn)
[11] Eisenhart L P 1917 *Trans. Am. Math. Soc.* **18** 111
[12] Eisenhart L P 1923 *Transformations of Surfaces* (Princeton, NJ: Princeton University Press) (reprinted 1962 (New York: Chelsea))
[13] Fouché M 1898 *Comp. Rend. Acad. Sci., Paris* **131** 210
[14] Jonas H 1915 *Berl. Math. Ges. Ber. Sitzungsber.* **14** 96
[15] Lamé G 1859 *Leçons sur la Théorie des Coordenées Curvilignes et leurs Diverses Applications* (Paris: Mallet-Bachalier)
[16] Lévy L 1886 *J. l École Polytecnique* **56** 63
[17] Liu Q P and Mañas M 1998 *J. Phys. A: Math. Gen.* **31** L193
[18] Liu Q P and Mañas M 1998 *J. Geom. Phys.* **27** 178
[19] Mañas M, Doliwa A and Santini P M 1997 *Phys. Lett.* A **232** 365
[20] Mañas M and Martínez Alonso L 1998 *Phys. Lett.* B **436** 316
[21] Petot A 1891 *Comp. Rend. Acad. Sci., Paris* **112** 1426

[22] Ribaucour A 1872 *Comp. Rend. Acad. Sci., Paris* **74** 1489

[23] Ribaucour A 1891 *J. Math. Pure Appl.* **7**

[24] Tsarev S P 1993 Classical differential geometry and integrability of systems of hydrodynamic type *Applications of Analytic and Geometrical Methods to Nonlinear Differential Equations* ed P A Clarkson (Dordrecht: Kluwer)